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Hilbert spaces of analytic functions and representations of the positive discrete series of Sp(6, ℝ)†

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Abstract. Hilbert spaces of analytic functions are constructed which carry irreducible representations of the positive discrete series of Sp(6, ℝ).

1. Introduction

Segal (1960) and Bargmann (1961) constructed a Hilbert space of analytic functions for the representations of the Weyl group associated with the oscillator creation and annihilation operators. This construction is closely related to the coherent states of the oscillator. In the present paper we generalise the approach taken by Bargmann to obtain a representation space for the symplectic group Sp(6, ℝ) (or Sp(3, ℝ) in mathematical notation) rather than the Weyl group.

Perelomov (1972) gave a general description of coherent states and constructed them in particular for representations of Sp(2, ℝ) (or Sp(1, ℝ) in mathematical notation). These coherent states have been applied to the theory of collective motion in nuclei (see, for example, Broeckhove *et al* 1984).

In Perelomov's construction, the coherent states are obtained by acting with unitary operators depending on group parameters on a normalised state of external weight in the representation space. Kramer and Saraceno (1981) studied coherent states obtained by acting with non-unitary operators depending analytically on complex parameters on the same states. For particular representations of Sp(6, ℝ), these analytic coherent states were constructed by Kramer (1982). The corresponding reproducing kernel and measure were already given by Hua (1963). The unitary coherent states with real parameters have also been constructed and applied to collective theories of nuclear closed shell configurations (see Kramer *et al* 1985a, b). Certain results on coherent states for representations of Sp(6, ℝ) corresponding to open shell dynamics were obtained by Filippov *et al* (1984) and Rowe (1984). Deenen and Quesne (1984) constructed for the same purpose so-called partial coherent states. In the present paper we construct analytic coherent states for general representations of Sp(6, ℝ) from the positive discrete series, and give the general method and explicit results for the corresponding reproducing kernels and measures. The results of the paper could be applied to collective theory for nuclear open shell configurations and to boson mappings for corresponding models (see Castaños *et al* 1985). After completion of this work we learned that Quesne has obtained very similar results.

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2. Hilbert spaces of analytic functions

Consider $\mathbb{C}^r$ with points $z=(z_1, z_2, \ldots, z_r)$ and functions $f: \mathbb{C}^r \to \mathbb{C}$ analytic in all complex variables z_j, $j=1,2,\ldots,r$. For the Hilbert space $\mathscr{H}$ we require
Condition ($\mathscr{H}$1). $\mathscr{H}$ has a scalar product $\langle | \rangle$ and a measure μ, $\mathscr{H}=\mathscr{L}^2(\mu)$ with

$$\langle f \mid q\rangle = \int \overline{f(z)}q(z)\,\mathrm{d}\mu(z)$$

$$\mathrm{d}\mu(z)=\rho(z,\bar{z})\prod_j \mathrm{dRe}(z_j)\,\mathrm{dIm}(z_j).$$

Condition ($\mathscr{H}$2). $\mathscr{H}$ has a reproducing kernel $I(z,z')$ with the property

$$f\in\mathscr{H}: f(z)=\int I(z,z')f(z')\,\mathrm{d}\mu(z').$$

We require that $\mathscr{H}$ carries a representation of a Lie group G with the following properties:
Condition (G1). $\mathscr{H}$ admits a Lie transformation group G,

$$\mathrm{G}=(g \mid g=g(\alpha_1,\alpha_2,\ldots,\alpha_s)g(0)=e)$$

such that under G

$$(z,g)\to z'=\phi(z,g) \qquad \phi(z,e)=z \qquad \phi \text{ analytic.}$$

Condition (G2). G acts on elements of $\mathscr{H}$ as

$$\begin{aligned}(T_g f)(z) &= \nu f(z')\\ &= \nu f(\phi(z,g)) \qquad \nu=\nu(z,g).\end{aligned}$$

Condition (G3). The representation of G is unitary.
To elaborate the conditions imposed on the Hilbert space by the conditions (G1, G2, G3) we consider the generators of G.

Definition 2.1. The generators X_i, $i=1,2,\ldots,s$ of G in $\mathscr{H}$ are the operators defined by

$$(\partial/\partial\alpha_i)(T_g f)(z)|_{\alpha=0}=(X_i f)(z) \qquad i=1,2,\ldots,s.$$

For the generators X_i we introduce the following notation:

$$X_i(z)=\sum_{j=1}^{r} f_{ij}(z)(\partial/\partial z_j)+c_i(z) \qquad i=1,2,\ldots,s.$$

For later use we define a second set of similar operators

$$\tilde{X}_i(z)=\sum_{j=1}^{r} f_{ij}(z)(\partial/\partial z_j)-c_i(z)+a_i(z)$$

$$a_i(z)=\sum_{j=1}^{r}(\partial/\partial z_j)f_{ij}(z).$$

The unitarity (G3) of the representation can now be expressed in terms of pairs $X_i(z)$, $(X^+)_i(z)$ which must be adjoint with respect to each other.

Proposition 2.2. Unitarity of the representation with respect to the reproducing kernel requires that the generators fulfil the differential equations

$$X_i(z)I(z,z')=\overline{(X^+)_i(z')I(z',z)} \qquad i=1,2,\ldots,s.$$

Proof. We write the scalar product in the form

$$\langle f\mid q\rangle=\int\int \overline{f(z)}I(z,z')q(z')\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(z')$$

and apply the conditions

$$\langle (X^+)_i f\mid q\rangle=\langle f\mid X_i q\rangle \qquad i=1,2,\ldots,s.$$

Proposition 2.3. Unitarity of the representation with respect to the measure μ requires that the weight function ρ for the measure fulfils the differential equations

$$\tilde{X}_i(z)\rho(z,\bar{z}')|_{z'=z}=\overline{(\tilde{X}^+)_i(z')\rho(z,\bar{z}')}|_{z'=z} \qquad i=1,2,\ldots,s.$$

Proof. We use the explicit form of the generators $X_i(z)$ given after definition 2.1, use the analytic property of the elements of $\mathcal{H}$ and perform a partial integration similar to the one considered by Bargmann (1961). For the functions f, q under consideration, we have to require that expressions

$$f_{ij}(z)q(z)\overline{f(z)}\rho(z,\bar{z}) \qquad i,j=1,2,\ldots,r$$

vanish at the boundary of the domain for the complex variables z. For the special cases in propositions 5.2 and 6.3 the boundary is defined as $|k_{ij}|\to 0$, $i,j=1,2,3$ and $(I-B^+B)\to 0$. The remaining conditions can be expressed in terms of the differential operators $\tilde{X}_i$ and yield the conditions stated above.

3. The complex form of Sp(6, ℝ) and analytic parameters for the cosets (U(1) × U(2))\Sp(6, ℝ), (U(1) × U(1) × U(1))\Sp(6, ℝ) and U(3)\Sp(6, ℝ)

As in Kramer (1982) we use the complex form of Sp(6, ℝ) given by

$$R\,\mathrm{Sp}(6,\mathbb{R})R^{-1}=\mathrm{Sp}(6,\mathbb{C})\cap \mathrm{U}(3,3) \qquad R=(1/2)^{1/2}\begin{vmatrix} I & \mathrm{i}I\\ I & -\mathrm{i}I\end{vmatrix}.$$

For $g\in \mathrm{Sp}(6,\mathbb{R})$ in complex form we define

$$g^{\S}=Mg^+M^{-1} \qquad M=\begin{vmatrix} I & 0\\ 0 & -I\end{vmatrix}$$

and

$$g^*=K{}^tgK^{-1} \qquad K=\begin{vmatrix} 0 & I\\ -I & 0\end{vmatrix}.$$

Then g is characterised by the two conditions

$$g^*=g^{-1} \qquad g^{\S}=g^{-1}.$$

We shall use the complex decomposition

$$g=\begin{vmatrix} I & 0\\ A & I\end{vmatrix}\begin{vmatrix} \lambda & 0\\ 0 & {}^t\lambda^{-1}\end{vmatrix}\begin{vmatrix} I & -B\\ 0 & I\end{vmatrix} \qquad A={}^tA,\ B={}^tB.$$

Now we state propositions on the analytic parametrisation of cosets which were examined by Kramer and Saraceno (1981) and Kramer (1982).

Proposition 3.1. The coset $(\mathrm{U}(1)\times\mathrm{U}(2))\backslash\mathrm{U}(3)$ admits the analytic parametrisation by the representatives

$$k=\begin{vmatrix}1 & k_{12} & k_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}\qquad k_{12}, k_{13} \text{ complex}$$

and has the real dimension $9-5=4$ (compare Kramer 1981, pp 63-70).

Proposition 3.2. The coset $\mathrm{U}(3)\backslash\mathrm{Sp}(6,\mathbb{R})$ admits the analytic parametrisation by the representatives

$$c'=\begin{vmatrix}I & -B\\ 0 & I\end{vmatrix}\qquad B={}^{t}B\qquad I-BB^{+}>0$$

or equivalently $I-B^{+}B>0$ (compare Kramer 1982).

Proposition 3.3. The coset $(\mathrm{U}(1)\times\mathrm{U}(2))\backslash\mathrm{Sp}(6,\mathbb{R})$ or $(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1))\backslash\mathrm{Sp}(6,\mathbb{R})$ admits the analytic parametrisation by the representatives

$$c=\begin{vmatrix}{}^{t}k^{-1} & 0\\ 0 & k\end{vmatrix}\begin{vmatrix}I & -B\\ 0 & I\end{vmatrix}$$

with k as given in proposition 3.1 or 6.1 and B as given in proposition 3.2. The dimension of this coset is $21-5=16$ or 18 corresponding to 8 or 9 complex parameters. The proof is given in the appendix.

We now examine the action of the group $\mathrm{Sp}(6,\mathbb{R})$ on the coset parametrised according to proposition 3.3. The 8 complex parameters k, B play the role of the complex numbers z considered in § 2, and the action to be considered determines the action of the transformation group $\mathrm{Sp}(6,\mathbb{R})$ according to (G1).

Proposition 3.4. The action of $\mathrm{Sp}(6,\mathbb{R})$ on the coset representative c from proposition 3.3. The map

$$\phi:\quad k'=\phi_1(k, B, \tilde{g})$$
$$B'=\phi_2(k, B, \tilde{g})$$

has the following properties. For ϕ_1 assume

$$k\rightarrow k\tilde{\lambda}=h'k'.$$

Then k' and h' are characterised by the equations

$$k'\colon k'_{1j}=[(k\tilde{\lambda})_{11}]^{-1}(k\tilde{\lambda})_{1j}\qquad j=1,2$$
$$h'\colon h'_{11}=(k\tilde{\lambda})_{11}$$
$$\det(h')=\det(k\tilde{\lambda})=\det\tilde{\lambda}.$$

In table 1 we give the maps ϕ_1, ϕ_2 for the three factors of general group element $\tilde{g}$.

Table 1. The action of Sp(6, ℝ) with elements g on the analytic parameters k, B.

$\tilde{g}$	ϕ_1	ϕ_2
$\begin{vmatrix} I & -\tilde{B} \\ 0 & I \end{vmatrix}$	$k'=k$	$B'=B+\tilde{B}$
$\begin{vmatrix} {}^t\tilde{\lambda}^{-1} & 0 \\ 0 & \tilde{\lambda} \end{vmatrix}$	$h'k'=k\tilde{\lambda}$	$B'={}^t\tilde{\lambda}B\tilde{\lambda}$
$\begin{vmatrix} I & 0 \\ \tilde{A} & I \end{vmatrix}$	$h'k'=k(I-\tilde{A}B)^{-1}$	$B'=B(I-\tilde{A}B)^{-1}$

4. The action of Sp(6, ℝ) on analytic functions

We now consider the positive discrete series representations of Sp(6, ℝ). These representations are characterised by an extremal weight and a corresponding extremal state. We shall consider the case where two weight components are equal

$$w=\{w_1w_2w_2\} \qquad w_1=h_1+(n/2) \qquad w_2=h_2+(n/2)$$

and the extremal state

$$|\mathrm{extr}\rangle = |(w_1w_2w_2)\rangle.$$

The case $w_1=w_2$ has been considered in Kramer (1982) and so we assume $w_1>w_2$ in what follows.

As in Kramer (1982) we shall assume that $|\mathrm{extr}\rangle$ is of highest weight with respect to U(3). Consider now an element of GL(3, ℂ) which is a product of a lower triangular matrix and a diagonal matrix and which we denote by h'. For the representation we then obtain

$$\tilde{g}=\begin{vmatrix} {}^th'^{-1} & 0 \\ 0 & h' \end{vmatrix} \qquad \langle\mathrm{extr}|T(\tilde{g})=(h'_{11})^{h_1-h_2}(\det h')^{w_2}\langle\mathrm{extr}|.$$

Definition 4.1. Let f be an element of the representation space of Sp(6, ℝ) characterised by the weight $w_1w_2w_2$. We define a map from f to an element of a Hilbert space of analytic functions by the prescription

$$f\to f(k,B)=\langle\mathrm{extr}|T(c)|f\rangle$$

with c given in proposition 3.3. Similarly we define the action of Sp(6, ℝ) on these functions by

$$(T_{\tilde{g}}f)(k,B)=\langle\mathrm{extr}|T(c\tilde{g})|f\rangle=\nu f(\phi_1(k,B,\tilde{g}),\phi_2(k,B,\tilde{g}))$$
$$\nu=(h'_{11})^{h_1-h_2}(\det h')^{w_2}.$$

Since $w_2=h_2+n/2$ can take half-integer values, we should give a prescription for the value of $(\det h')^{w_2}$ in this expression. We shall make use of Bargmann's prescription (Bargmann 1968) for half-integer powers of this determinant. This prescription arises from the consideration of the universal covering group of the real symplectic group.

With this definition we are now ready to construct the generators according to definition 2.1. We give the defining relations for these generators in table 2.

Table 2. Definition of the generators of $\mathrm{Sp}(6, \mathbb{R})$ for elements according to table 1.

$\tilde{g}$	$(\partial/\partial\alpha_i)(T_g f)(k, B)\|_{\alpha=0}$	X_i
$\begin{vmatrix} I & -\tilde{B} \\ 0 & I \end{vmatrix}$	$\frac{1}{2}((\partial/\partial\tilde{b}_{ij}) + (\partial/\partial\tilde{b}_{ji}))(T_g f)\|_{\tilde{B}=0}$	$K_{ij,-}$
$\begin{vmatrix} {}^t\tilde{\lambda}^{-1} & 0 \\ 0 & \tilde{\lambda} \end{vmatrix}$	$(\partial/\partial\tilde{\theta}_{ij})(T_g f)\|_{\theta=0}$	C_{ij}
$\tilde{\lambda} = I + \tilde{\theta}$		
$\begin{vmatrix} I & 0 \\ \tilde{A} & 0 \end{vmatrix}$	$\frac{1}{2}((\partial/\partial\tilde{a}_{ij}) + (\partial/\partial\tilde{a}_{ij}))(T_g f)\|_{\tilde{A}=0}$	$K_{ij,+}$

Proposition 4.2. The generators of $\mathrm{Sp}(6, \mathbb{R})$ with respect to analytic functions of (k, B) have the explicit form

$$K_{ij,-} = K_{ij,-}(B) = \Delta_{ij}$$

$$\Delta_{ij} = \tfrac{1}{2}((\partial/\partial b_{ij}) + (\partial/\partial b_{ji}))$$

$$C_{ij} = C_{ij}(k) + C_{ij}(B)$$

$$C_{ij}(k) = D_{ij}(k) + (h_1 - h_2)F_{ij} + w_2\delta_{ij}$$

$$D(k) = \begin{bmatrix} -k_{12}(\partial/\partial k_{12}) - k_{12}(\partial/\partial k_{13}) & (\partial/\partial k_{12}) & (\partial/\partial k_{13}) \\ -(k_{12})^2(\partial/\partial k_{12}) - k_{12}k_{13}(\partial/\partial k_{13}) & k_{12}(\partial/\partial k_{12}) & k_{12}(\partial/\partial k_{13}) \\ -k_{12}k_{13}(\partial/\partial k_{12}) - (k_{13})^2(\partial/\partial k_{13}) & k_{13}(\partial/\partial k_{12}) & k_{13}(\partial/\partial k_{13}) \end{bmatrix}$$

$$F = \begin{vmatrix} 1 & 0 & 0 \\ k_{12} & 0 & 0 \\ k_{13} & 0 & 0 \end{vmatrix}$$

$$C_{ij}(B) = 2\sum_r b_{ir}\Delta_{jr}$$

$$K_{ij,+} = K_{ij,+}(k) + \tfrac{1}{2}\sum_m (b_{jm}C_{im}(k) + b_{im}C_{jm}(k))$$

$$K_{ij,+}(B) = \sum_{r,s} b_{ir}b_{js}\Delta_{rs}.$$

The symmetrised derivatives have been introduced since we wish to apply these operators to functions of the matrix B irrespective of the order of the matrix indices.

5. The reproducing kernel and the measure

Having found the expressions for the generators, the reproducing kernel and the measure should be found as the solutions of the differential equations given in propositions 2.2 and 2.3. Fortunately it is possible to derive the form of the reproducing kernel from the group multiplication law. For this purpose we define the element

$$c(k, B)c^{\S}(k', B') = \begin{vmatrix} \lambda & * \\ * & * \end{vmatrix}$$

where the other blocks are not needed. Then we get from definition 4.1 the expression

$$\begin{aligned}\langle k, B|k', B'\rangle &= \langle \text{extr}|T(c(k, B))T(c^{\S}(k', B'))|\text{extr}\rangle \\ &= (\Delta_1^1)^{h_1-h_2}(\Delta_{123}^{123})^{w_2}\end{aligned}$$

where the Δ are subdeterminants of the matrix ${}^t\lambda^{-1}$. From the explicit expressions for c and $c^{\S}$ we derive

$$\begin{aligned}&\Delta_1^1 = (kVk'^{+})_{11} \\ &\Delta_{123}^{123} = \det V \\ &V = (I - B'^{+}B)^{-1}.\end{aligned}$$

Proposition 5.1. The expression $\langle k, B|k', B'\rangle$ given above fulfils with respect to the generators the differential equations of proposition 2.2 for the reproducing kernel I with the adjoint relations

$$\begin{aligned}&(K_{ij,-})^{+} = K_{ij,+} \\ &(C_{ij})^{+} = C_{ji}\end{aligned}$$

and therefore provides the reproducing kernel for a Hilbert space $\mathscr{H}$ of analytic functions.

Proof. Denote an abstract generator of the group in the representation space by $\hat{X}_i$. Then by use of proposition 2.2 we have

$$X_i(z)\langle z|z'\rangle = \langle z|\hat{X}_i|z'\rangle.$$

Using this equation and its adjoint it is easy to show that the overlap indeed fulfils the differential equation required in proposition 2.2 for the reproducing kernel.

We turn now to the measure and according to the proposition have to construct the differential operators $\tilde{X}_i$. An explicit computation, taking into account the symmetrised derivatives, yields the following expressions:

$$\begin{aligned}&\tilde{K}_{ij,-} = K_{ij,-} \\ &\tilde{C}_{ij}(k) = D_{ij}(k) + (-h_1 + h_2 + 3)F_{ij} + (-w_2 + 1)\delta_{ij} \\ &\tilde{C}_{ij}(B) = C_{ij}(B) + 4\delta_{ij} \\ &\tilde{K}_{ij,+} = \tfrac{1}{2}\sum_m (b_{jm}\tilde{C}_{im}(k) + b_{im}\tilde{C}_{jm}(k)) + \tilde{K}_{ij,+}(B) \\ &\tilde{K}_{ij,+}(B) = K_{ij,+}(B) + 4b_{ij}.\end{aligned}$$

Proposition 5.2. Consider the generators of $\mathrm{Sp}(6, \mathbb{R})$ in the form

$$X_i = X_i(k, B, h_1 - h_2, w_2).$$

Then the differential operators $\tilde{X}_i$ may be expressed as

$$\tilde{X}_i(k, B, h_1 - h_2, w_2) = X_i(k, B, -h_1 + h_2 + 3, -w_2 + 5).$$

Therefore the differential equations for the measure given in proposition 2.3 are solved by the expression

$$\rho(k, B, \bar{k}', \bar{B}') = c_0(\Delta_1^1)^{-h_1+h_2+3}(\Delta_{123}^{123})^{-w_2+5}$$

and the weight function for the measure is given by

$$\rho = \rho(k, B, \bar{k}, \bar{B}).$$

6. Representations and Hilbert space for a general weight

The analysis follows the same pattern as before, and we give only the key results.

Proposition 6.1. The matrix k for the coset $(\mathrm{U}(1)\times \mathrm{U}(1)\times \mathrm{U}(1))\backslash \mathrm{U}(3)$ has the form

$$k = \begin{vmatrix} 1 & k_{12} & k_{13} \\ 0 & 1 & k_{23} \\ 0 & 0 & 1 \end{vmatrix} \qquad k_{12}, k_{13}, k_{23} \text{ complex.}$$

Proposition 6.2. The generators of $\mathrm{Sp}(6, \mathbb{R})$ with respect to the complex variables (k, B) have the same general form as given in proposition 4.2 but now with $C_{ij}(k)$ given by

$$C_{ij}(k) = D_{ij}(k) + (h_1 - h_2)F_{ij}^1(k) + (h_2 - h_3)F_{ij}^2(k) + w_3\delta_{ij}$$

where

$$w_i = h_i + (n/2) \qquad h_1 > h_2 > h_3$$

$$D(k) = \begin{bmatrix} -k_{12}(\partial/\partial k_{12}) - k_{13}(\partial/\partial k_{13}) & (\partial/\partial k_{12}) & (\partial/\partial k_{13}) \\ -k_{12}(\partial/\partial k_{12}) - k_{13}k_{12}(\partial/\partial k_{13}) - (k_1\times k_2)_1(\partial/\partial k_{23}) & k_{12}(\partial/\partial k_{12}) - k_{23}(\partial/\partial k_{23}) & k_{12}(\partial/\partial k_{13}) + (\partial/\partial k_{23}) \\ -k_{12}k_{13}(\partial/\partial k_{12}) - k_{13}(\partial/\partial k_{13}) - (k_1\times k_2)_1 k_{23}(\partial/\partial k_{23}) & k_{13}(\partial/\partial k_{12}) - k_{23}(\partial/\partial k_{23}) & k_{13}(\partial/\partial k_{13}) + k_{23}(\partial/\partial k_{23}) \end{bmatrix}$$

$$F^1(k) = \begin{vmatrix} 1 & 0 & 0 \\ k_{12} & 0 & 0 \\ k_{13} & 0 & 0 \end{vmatrix} \qquad F^2(k) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (k_1\times k_2)_1 & k_{23} & 0 \end{vmatrix}$$

and $(k_1\times k_2)_1 = k_{13} - k_{12}k_{23}$.

Proposition 6.3. The reproducing kernel for a general weight is given by

$$\langle kB|k'B'\rangle = (\Delta_1^1)^{h_1-h_2}(\Delta_{12}^{12})^{h_2-h_3}(\Delta_{123}^{123})^{w_3}$$

where the subdeterminants are taken from the matrix

$$k(I-(B')^+B)^{-1}(k')^+.$$

The measure is given by

$$\rho(kB, k'B') = (\Delta_1^1)^{h_2-h_1-2}(\Delta_{12}^{12})^{h_3-h_2-2}(\Delta_{123}^{123})^{-w_3+6}$$

at the value $k'=k$, $B'=B$.

Appendix. Proof of proposition 3.3

We give here the proof for the case where $w_1 \neq w_2 \neq w_3 \neq w_1$. We have to show that the coset $(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1))\backslash\mathrm{Sp}(6,\mathbb{R})$ can be parametrised by the complex numbers B and k. We first consider the complex extension $\mathrm{Sp}(6,\mathbb{C})$ of the symplectic group. According to the relations given after definition 2.1 the real symplectic group is obtained by the restriction to $\mathrm{U}(3,3)$. The group $\mathrm{Sp}(6,\mathbb{C})$ admits the block decomposition

$$g = \begin{vmatrix} I & 0 \\ A & I \end{vmatrix} \begin{vmatrix} \lambda & 0 \\ 0 & {}^t\lambda^{-1} \end{vmatrix} \begin{vmatrix} I & -B \\ 0 & I \end{vmatrix} \tag{A1}$$

where A and B are complex symmetric matrices and $\lambda \in \mathrm{GL}(3,\mathbb{C})$. By the well known Gauss decomposition, λ or ${}^t\lambda^{-1}$ admits the factorisation

$${}^t\lambda^{-1} = \Delta_-\Delta_0 k \tag{A2}$$

where Δ_- is a lower triangular matrix, Δ_0 is a diagonal matrix and k is an upper triangular matrix. We use this factorisation to pass from the decomposition of g (equation (A1)) to

$$g = \begin{vmatrix} \Delta_0^{-1} & 0 \\ 0 & \Delta_0 \end{vmatrix} \begin{vmatrix} {}^t(\Delta'_-)^{-1} & 0 \\ 0 & \Delta'_- \end{vmatrix} \begin{vmatrix} I & 0 \\ A' & I \end{vmatrix} \begin{vmatrix} {}^tk^{-1} & 0 \\ 0 & k \end{vmatrix} \begin{vmatrix} I & -B \\ 0 & I \end{vmatrix} \tag{A3}$$

where the lower triangle matrix Δ'_- is defined as

$$\Delta'_- := \Delta_0^{-1}\Delta_-\Delta_0 \tag{A4}$$

and the symmetric complex matrix A' as

$$A' := (\Delta_-\Delta_0)^{-1}A^t(\Delta_-\Delta_0)^{-1}. \tag{A5}$$

The diagonal matrix Δ_0 has a unique decomposition $\Delta_0 = h\tilde{\Delta}_0$ into a real positive diagonal matrix $\tilde{\Delta}_0$ times an element h of the group $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$. By dropping the element h and defining

$$l = \tilde{\Delta}_0\Delta'_- \tag{A6}$$

we obtain a unique characterisation of the representative g_c for the coset$(\mathrm{U}(1)\times\mathrm{U}(1)\times \mathrm{U}(1))\backslash\mathrm{Sp}(6,\mathbb{C})$ of the form

$$g_c = \begin{vmatrix} {}^tl^{-1} & 0 \\ 0 & l \end{vmatrix} \begin{vmatrix} I & 0 \\ A' & I \end{vmatrix} \begin{vmatrix} {}^tk^{-1} & 0 \\ 0 & k \end{vmatrix} \begin{vmatrix} I & -B \\ 0 & I \end{vmatrix}. \tag{A7}$$

To pass from equation (A7) to a coset representative for $(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1))\backslash\mathrm{Sp}(6,\mathbb{R})$, we now restrict g_c to the group U(3, 3) which implies

$$g_c g_c^{\S} = g_c^{\S} g_c = I \tag{A8}$$

where

$$g_c^{\S} = M g_c^{+} M \qquad M = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}. \tag{A9}$$

In terms of the matrix blocks B, k, l and A', these conditions are equivalent to the two equations

$$l^{+} l = (k^{+})^{-1}(I - B^{+}B)k^{-1} \tag{A10}$$

$$A' = -(l^{+}l)^{-1}(k^{-1})^{+}B^{+}\,{}^{t}k. \tag{A11}$$

Now we consider the parameters B, k as independent complex parameters subject only to the restrictions on their forms and for B to ${}^{t}B = B$, $I - B^{+}B > 0$. It can easily be verified by an explicit computation that the set of equations (A10) has a unique non-analytic solution $l = l(k, B)$ which, when inserted into equation (A11), yields a corresponding solution for $A' = A'(k, B)$. The matrix g_c of equation (A7) restricted in this fashion yields now the coset representatives of $(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1))\backslash\mathrm{Sp}(6,\mathbb{R})$. We stress that the construction of analytic coherent states in definition 4.1 requires that we do not use the non-analytic first two factors of g_c.

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